ELSEVIER
SCIENCE DIRECT®
PHYSICS LETTERS B
Physics Letters B 575 (2003) 137–143
www.elsevier.com/locate/physletb

On the time-dependent oscillator and the nonlinear realizations of the Virasoro group

V.P. Akulov [a,b], Sultan Catto [b,c], Oktay Cebecioğlu [d], A. Pashnev [e]

[a] *Department of Natural Sciences, Baruch College of the City University of New York, New York, NY 10010, USA*
[b] *Physics Department, The Graduate School and University Center, The City University of New York, 365 Fifth Ave, New York, NY 10016-4309, USA*
[c] *Center for Theoretical Physics, The Rockefeller University, 1230 York Ave, New York, NY 10021-6399, USA*
[d] *Department of Physics, Kocaeli University, Atatürk Bulvarı, Anıtpark Yanı, Kocaeli, Turkey*
[e] *Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia*

Received 18 June 2003; received in revised form 9 September 2003; accepted 11 September 2003

Editor: M. Cvetič

Abstract

Using the nonlinear realizations of the Virasoro group we construct the action of the conformal quantum mechanics (CQM) with additional harmonic potential. We show that $SL(2,R)$ invariance group of this action is nontrivially embedded in the reparametrization group of the time which is isomorphic to the centerless Virasoro group. We generalize the consideration to the Ermakov systems and construct the action for the time-dependent oscillator. Its symmetry group is also the $SL(2,R) \sim SU(1,1)$ group embedded in the Virasoro group in a more complicated way.

1. Introduction

The time-dependent oscillator (the so-called Ermakov system [1]) has been the subject of vigorous research for decades because of its relevance to a large variety of physical phenomena, especially due to its elegant mathematical properties and application potential of its invariant. The list of works is very exhaustive but is merely a small part of the amount of them all. It includes, in particular, the study of soliton solutions of nonlinear evolution equations [2], construction of time-dependent integrals of motion for the parametric harmonic oscillator used for the canonical formulation of more general parametric systems and their semiclassical quantization [3], theory of coherent and squeezed states [4], Berry's phase [5], Noether's theorem and Noether and Lie symmetries of the time-dependent Kepler system [6], anisotropic Bose–Einstein condensates and completely integrable dynamical systems [7], cosmological particle creation [8], scalar field cosmologies and inflationary scenarios [9], nonlinear optics [10], propagation of water waves [11], exact solution for the Calogero system [12] and noncentral potential with dynamic symmetry [13], study of motions in a Paul trap [14], quantum mechanical description of highly cooled atoms [15], emergence of non-classical optical states of light due to time-dependent dielectric con-

E-mail addresses: akulov@juno.com (V.P. Akulov), scatto@gc.cuny.edu (S. Catto), ocebecioglu@yahoo.com (O. Cebecioğlu), pashnev@thsun1.jinr.ru (A. Pashnev).

0370-2693 © 2003 Elsevier B.V. Open access under CC BY license.
doi:10.1016/j.physletb.2003.09.044

stant [16], particle distribution for beam in electric field [17], nonlinear elasticity [18], molecular structures [19], quantum field theory in curved spaces [20], quantum cosmology [21], etc., among others.

A vital modification of the time-dependent oscillator includes an additional term in the potential proportional to the inverse square of the coordinate—it is often referred to as the anharmonic oscillator. This extra term is conformally invariant. The analogous conformal quantum mechanics (CQM) was investigated in detail by De Alfaro, Fubini and Furlan [22]. It was shown in their paper that the consistent quantum treatment of the model assumes the transition to the new time coordinate, which transpires to be equivalent to the introduction of additional oscillator-like term with a constant frequency in the potential. Therefore, the emerging physical Hamiltonian represents the anharmonic oscillator with time-independent frequency ω.

Another fascinating feature of conformal quantum mechanics (CQM) [22], as well as its supersymmetric generalization—SCQM [23–32], is the fact that they are the simplest theories that allow the cultivation and development of methods for investigation of more complicated higher-dimensional field theories. One should also note that in spite of its simplicity, SCQM describes the physical objects such as a particle near the horizon of black hole [34], etc. The extended SCQM is also closely related to the Calogero model with spin, which has numerous physical applications.

The most adequate approach for understanding the geometrical meaning of CQM and SCQM is the method of nonlinear realizations of the symmetry groups underlying both theories—the group $SL(2,R)$ and its supersymmetrical generalizations $SU(1,1|1)$ and $SU(1,1|2)$, respectively [33], [26]. In spite of its power, this method does not allow the possibility of including in the Hamiltonian of the theory the oscillator-like potentials introduced in [22]. As will be shown in this Letter the explanation for this lies entirely in the fact that in the presence of the oscillator-like term the invariance group of the action, though being the conformal group, is realized by the more complicated transformations. These transformations for the constant ω, as well as for the time-dependent one (Ermakov system), can naturally be embedded in the reparametrization group of the time variable which is isomorphic to the centerless Virasoro group. This embedding is rather nontrivial in the case of nonvanishing ω.

The structure of the Letter is as follows. In Section 2 we apply the nonlinear realizations method to the Virasoro group and its three-dimensional subgroup $SL(2,R)$. We calculate the transformation laws for parameters of these groups and construct differential Cartan's Omega-forms invariant under these transformations. They are then used for the construction of the action for conformal quantum mechanics in the Subsection 3.1. In the Subsection 3.2 we illustrate the mechanism for the appearance of the oscillator-like terms in the Omega-forms, and correspondingly in the action. We show how the symmetry group of this action, $SL(2,R)$, is nontrivially embedded in the Virasoro group. In Subsection 3.3 we generalize these results to the Ermakov systems with time-dependent oscillator frequency. We describe also the transformations of the time and phase space variables which connect with each other the Hamiltonians with different values of the harmonic oscillator frequency, including the free motion and Ermakov systems. Some further anticipations of the formalism we developed are included in conclusions.

2. The nonlinear realization of the reparametrization group

The generators of the infinite-dimensional reparametrization (diffeomorphisms) group on the line parametrized by some parameter s are $L_m = i s^{m+1}\frac{d}{ds}$ and form the Virasoro algebra without central charge

$$[L_n, L_m] = -i(n-m)L_{n+m}. \tag{2.1}$$

If one restricts to the regular at the origin $s=0$ transformations, it is convenient to parametrize the Virasoro group element as [35,36]

$$G = e^{i\tau L_{-1}} \cdot e^{ix_1L_1} \cdot e^{ix_2L_2} \cdot e^{ix_3L_3} \cdots e^{ix_0L_0}, \tag{2.2}$$

where all multipliers, except the last one, are arranged by the conformal weight of the generators in the exponents.

The transformation laws of the group parameters τ, x_n in (2.2) under the left action

$$G' = (1+ia)G, \tag{2.3}$$

where infinitesimal element a belongs to the Virasoro algebra

$$a = a_0 L_{-1} + a_1 L_0 + a_2 L_1 + \cdots + a_m L_m + \cdots = \sum_{n=0}^{\infty} a_n L_{n-1}, \tag{2.4}$$

are

$$\delta\tau = a(\tau), \tag{2.5}$$

$$\delta x_0 = \dot{a}(\tau), \tag{2.6}$$

$$\delta x_1 = -\dot{a}(\tau) x_1 + \frac{1}{2}\ddot{a}(\tau), \tag{2.7}$$

$$\delta x_2 = -2\dot{a}(\tau) x_2 + \frac{1}{6}\dddot{a}(\tau), \tag{2.8}$$

$$\ldots,$$

where the infinitesimal function $a(\tau)$ is constructed out of the parameters a_n

$$a(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + \cdots = \sum_{n=0}^{\infty} a_n \tau^n. \tag{2.9}$$

One can see from (2.5) that the parameter τ transforms precisely as the coordinate of the one-dimensional space under the reparametrization. The parameters x_0 and x_1 transform correspondingly as the dilaton and one-dimensional Cristoffel symbol. In general the transformation rule for x_n contains $(n+1)$st derivative of the infinitesimal parameter $a(\tau)$.

To make the connection with the physical models, it is natural to consider all parameters x_n, $n = 0, 1, 2, \ldots$, in (2.2) as the fields in one-dimensional space parametrized by the coordinate τ.

The conformal group $SL(2, R) \sim SU(1, 1)$ in one dimension is a three-parameters subgroup of (2.2), namely, the one generated by L_{-1}, L_0 and L_1. Its group element is a product of first two and last one multipliers in the expression (2.2)

$$G_C = e^{i\tau L_{-1}} \cdot e^{ix_1 L_1} \cdot e^{ix_0 L_0}. \tag{2.10}$$

In other words, the $SL(2, R)$ group is embedded in the Virasoro group in the simplest way by the conditions

$$x_n = 0, \quad n \geqslant 2. \tag{2.11}$$

The infinitesimal transformation function $a(\tau)$ (2.4) which conserves the conditions (2.11) contains only three parameters

$$a(\tau) = a_0 + a_1\tau + a_2\tau^2. \tag{2.12}$$

It is convenient to introduce new variables playing the roles of the coordinate and momentum of the particle

$$x = e^{x_0/2}, \qquad p = x_1 x, \tag{2.13}$$

for which the conformal group infinitesimal transformations are

$$\delta\tau = a(\tau), \tag{2.14}$$

$$\delta x = \frac{1}{2}\dot{a}(\tau) x, \tag{2.15}$$

$$\delta p = -\frac{1}{2}\dot{a}(\tau) p + \frac{1}{2}\ddot{a}(\tau) x, \tag{2.16}$$

with $a(\tau)$ given in this case by the expression (2.12). The corresponding finite transformations are

$$\tau' = \frac{a\tau + b}{c\tau + d}, \qquad x' = \frac{x}{c\tau + d}, \qquad p' = (c\tau + d)p - cx, \tag{2.17}$$

with parameters of the transformation constrained by the unimodularity condition $ad - bc = 1$.

3. The application of nonlinear realizations of the $SL(2, R)$ group for the actions construction

3.1. The action integral for conformal mechanics

The transformations (2.17) form a symmetry group of the conformal quantum mechanics of [22] with the action

$$S = \frac{1}{2}\int d\tau \left(\dot{x}^2 - \frac{\gamma}{x^2}\right). \tag{3.1}$$

As was shown in [33] (see also [31]) this action can be naturally described on the language of invariant differential Cartan's form

$$\Omega_C = G_C^{-1}\, dG_C = \Omega_{-1} L_{-1} + \Omega_0 L_0 + \Omega_1 L_1 \tag{3.2}$$

connected with the parametrization (2.10) of the conformal group. The explicit calculations give

$$\Omega_{-1} = \frac{d\tau}{x^2}, \tag{3.3}$$

$$\Omega_0 = \frac{dx - p\,d\tau}{x}, \tag{3.4}$$

$$\Omega_1 = x\,dp - p\,dx + p^2\,d\tau. \tag{3.5}$$

All these differential forms are invariant under the transformations (2.17) and can be used for construction of an invariant action. The simplest one is the linear in Ω-forms combination [33]

$$S = -\frac{1}{2}\int \omega_1 - \frac{\gamma}{2}\int \omega_{-1} = \int d\tau \left(-\frac{1}{2}\left(x\dot{p} - p\dot{x} + p^2\right) - \frac{\gamma}{x^2}\right). \tag{3.6}$$

The first term in this expression is appropriately normalized to get the correct kinetic term. The parameter γ plays the role of cosmological constant in one dimension because ω_{-1}, which corresponds to the translation generator L_{-1}, is the differential 1-form einbein.

The action (3.6) is a first-order representation of the action describing the conformal mechanics of De Alfaro, Fubini and Furlan [22]. Indeed, one can find p by solving its equation of motion, insert it back in the Lagrangian and get the second-order action (3.1).

3.2. The action integral for conformal mechanics with additional harmonic potential

From the point of view of underlying physics the action (3.1) is not satisfactory one, because the corresponding quantum mechanical Hamiltonian does not have the ground state. The modification of this action with the appealing spectrum of the energy was considered in [22]. It includes the additional harmonic oscillator term

$$S_1 = \frac{1}{2}\int d\tau \left(\dot{x}^2 - \frac{\gamma}{x^2} - \omega^2 x^2\right). \tag{3.7}$$

Though the action (3.7) contains the dimensional parameter ω, it is invariant under the transformations of conformal group, realized by the more complicated expressions, as we will see.

As we have already mentioned in the Introduction, the action (3.7) cannot be described in the framework of nonlinear realizations of the $SL(2,R)$ group, parametrized as in (2.10). Instead, we will consider the embedding of this group in the Virasoro group (2.2) by conditions different from the simplest ones (2.11). The structure of the component Ω_1^V in the Cartan's Omega-form connected with the Virasoro group

$$\Omega_V = G^{-1}\,dG = \Omega_{-1}L_{-1} + \Omega_0 L_0 + \Omega_1^V L_1 + \Omega_2^V L_2 + \cdots \tag{3.8}$$

may serve as a hint in the choice of the appropriate conditions. The components Ω_{-1} and Ω_0 coincide with the corresponding components (3.3) and (3.4). Though the components $\Omega_2^V, \Omega_3^V, \ldots$ depend in general on all parameters x_n, the component Ω_1^V

$$\Omega_1^V = x\,dp - p\,dx + p^2\,d\tau - 3x_2 x^2\,d\tau \tag{3.9}$$

depends in addition to the phase space variables (x,p) only on the parameter $x_2 = x_2(\tau)$. So, the last term in expression (3.9) is the only difference of it with respect to the corresponding expression (3.5) calculated for the representation (2.2) of the $SL(2,R)$ group. Moreover, if we take

$$x_2(\tau) = -\frac{1}{3}\omega^2, \quad \omega = \text{const}, \tag{3.10}$$

we obtain exactly an oscillator-like term in the action

$$S = -\frac{1}{2}\int \omega_1 - \frac{\gamma}{2}\int \omega_{-1} = \int d\tau \left(-\frac{1}{2}\left(x\dot{p} - p\dot{x} + p^2 + \omega^2 x^2\right) - \frac{\gamma}{x^2}\right), \tag{3.11}$$

which coincides with the action S_1 (3.7) in the second-order form.

The component ω_1 (3.9) by construction is invariant under the arbitrary infinitesimal transformations (2.3) of the Virasoro group if parameters x, p, and x_2 transform according to Eqs. (2.15), (2.16) and (2.8). The consistency condition of this last transformation law with the demand that $\omega = \text{const}$ can be written in the form

$$\dddot{a}(\tau) + 4\omega^2\dot{a}(\tau) = 0. \tag{3.12}$$

The solution of this differential equation gives

$$a(\tau) = a_0 + a_1\sin(2\omega\tau) + a_2\cos(2\omega\tau). \tag{3.13}$$

So, the action of conformal mechanics (3.7) with additional oscillator-like potential is invariant under the three parameter transformation (3.13).

3.3. The time-dependent oscillator

In general the variable $x_2(\tau)$ can be arbitrary function of the time. Nevertheless, the treating it as a dynamical variable leads to the trivial dynamics because, as one can easily see from expression (3.9), it plays the role of a Lagrange multiplier leading to the equation of motion $x^2 = 0$.[1] So, instead of being the constant as in a previous subsection, the parameter x_2 in a physical model can be at most some *fixed* function $x_2(\tau)$. It means that after the time transformation (2.5) $\tau \to \tau' = \tau + a(\tau)$ the functional dependence remains the same: $x_2(\tau) \to x_2(\tau')$, $\delta x_2(\tau) = a(\tau)\dot{x}_2(\tau)$. Introducing the time-dependent frequency by the relation $x_2(\tau) = -\omega^2(\tau)/3$, the transformation law (2.8) leads to the equation for the infinitesimal parameter $a(\tau)$

$$\dddot{a}(\tau) + 4\omega^2(\tau)\dot{a}(\tau) + 2\frac{d}{d\tau}\big(\omega^2(\tau)\big)a(\tau) = 0. \tag{3.14}$$

This differential equation of the third order with the time-dependent coefficients has the solution in the form [39]

$$a(\tau) = C_1 u_1^2 + C_2 u_1 u_2 + C_3 u_2^2, \tag{3.15}$$

where C_1, C_2, C_3 are three infinitesimal constants and functions $u_1(\tau), u_2(\tau)$ form the fundamental system of solutions of auxiliary equation

$$\ddot{u}(\tau) + \omega^2(\tau)u(\tau) = 0. \tag{3.16}$$

For the time-independent ω this solution reproduces the ones given by (3.13). For different particular forms of the $\omega^2(\tau)$ Eq. (3.14) becomes, for example, the Lamé, Matieu, Hill, etc., equations [39], each of which play the very important role in the physics.

So, the solution (3.15) of Eq. (3.14) describe the invariance transformations of the action for the time-dependent oscillator with the frequency $\omega(\tau)$.

The very important for the time-dependent oscillator model is the question about its possible connection with other solvable systems like harmonic oscillator or even with the free particle. Such connection can in principle lead to the construction of exact solutions of the Schrödinger equation for the time-dependent oscillator starting from the solutions of these simpler systems. The example of such connection was given in [22] where the transformation from the system with vanishing frequency (3.1) to the one with constant ω (3.7) was given. This transformation inavitably includes the transformation of the time.

To construct the generalizations of this transformation let us consider the most general finite transformation of the Virasoro group [31]

$$\tau \to \tau' = f(\tau), \tag{3.17}$$

$$x(\tau) \to x'(\tau') = \big(\dot{f}(\tau)\big)^{1/2} x(\tau), \tag{3.18}$$

$$p(\tau) \to p'(\tau') = \frac{1}{(\dot{f}(\tau))^{1/2}} p(\tau) + \frac{\ddot{f}(\tau)}{2(\dot{f}(\tau))^{3/2}} x(\tau), \tag{3.19}$$

$$x_2(\tau) \to x_2'(\tau') = \frac{1}{(\dot{f}(\tau))^2} x_2(\tau) + \frac{1}{2}\frac{d}{d\tau}\left(\frac{\ddot{f}(\tau)}{\dot{f}(\tau)}\right) - \frac{1}{4}\left(\frac{\ddot{f}(\tau)}{\dot{f}(\tau)}\right)^2. \tag{3.20}$$

If we start with $x_2(\tau) = 0$, i.e., in the absence of the oscillator like potential, the frequency of the induced harmonic term in the new time will be given by the expression

$$\omega^2(\tau) = \frac{1}{2}\frac{d}{d\tau}\left(\frac{\ddot{f}(\tau)}{\dot{f}(\tau)}\right) - \frac{1}{4}\left(\frac{\ddot{f}(\tau)}{\dot{f}(\tau)}\right)^2, \tag{3.21}$$

which can be recognized as the Schwarz derivative of the transformed time over the old ones. One can rewrite Eq. (3.21) in the more familiar form

$$\left(\frac{d^2}{d\tau^2} + \omega^2(\tau)\right)\frac{1}{\sqrt{\dot{f}(\tau)}} = 0. \tag{3.22}$$

So, the solution $f(\tau)$ of Eq. (3.22) gives the transformation rules (3.17)–(3.20) between the two systems—the one without oscillator-like potential and the others having in addition to the conformal potential $\sim 1/x^2$ the term $\omega^2(\tau)x^2$. One should note also that the conformal potential term in the action is by itself invariant under the arbitrary finite transformations (3.17), (3.18).

4. Conclusions

In this Letter we applied the methods of nonlinear realizations approach for construction of the actions

[1] If the variable x carries in addition some index I, $x \to x_I$, the situation drastically changes when this index describes the vector representation of the rotation group of the space–time with the signature $(D, 2)$. In this case the action is given by the sum of $D + 2$ expressions (3.9) (with the corresponding signs) and it describes the massless particle in D-dimensional space–time [37] (the spinning particle if instead of the Virasoro group one considers the reparametrization group in the superspace $(1, N)$ with one bosonic and N Grassmann coordinates [38]).

of conformal quantum mechanics, as well, as the action of the time-dependent oscillator. We have shown that both the actions are invariant under the three parameter transformations which are nontrivially embedded in the Virasoro group. We described also different types of transformations belonging to the Virasoro group and making the connections of different systems each with other. In particular, we obtain such transformation between the free motion Hamiltonian and Hamiltonian of the time-dependent oscillator. It would be interesting to carry up the analogous considerations in the more complicated theories, such as $N = 2$ and $N = 4$ superconformal quantum mechanics.

Acknowledgements

A.P. thanks the members of Graduate School and University Center, the City University of New York, where the essential part of this work was done. The work of A.P. was partially supported by INTAS grant, project No. 00-00254.

References

[1] V. Ermakov, Univ. Izv. Kiev Serie III 9 (1880) 1.
[2] M.V. Ioeffe, H.J. Korsch, math-ph/0209013;
F. Calogero, A. Degasperis, Spectral Transform and Solitons, Vol. 1, North-Holland, Amsterdam, 1982;
V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer-Verlag, Berlin, 1981.
[3] H.R. Lewis, W.E. Lawrence, Phys. Rev. A 55 (1997) 2615;
H. Mostafazadeh, Phys. Rev. A 55 (1977) 4084;
H. Mostafazadeh, quant-ph/9807002;
K.E. Thylwe, H. Dankowicz, J. Phys. A: Math. Gen. 29 (1996) 3707;
K.E. Thylwe, H. Dankowicz, J. Phys. A: Math. Gen. 31 (1998) 2253;
D.Y. Song, quant-ph/9812038.
[4] V.I. Manko, in: N.M. Atakishiev (Ed.), Proc. IV Wigner Symp., World Scientific, Singapore, 1996;
R.S. Kaushal, D. Parashar, Phys. Rev. A 55 (1997) 2610;
J.Y. Yi, J.K. Kim, K.S. Soh, Phys. Rev. A 51 (1995) 4268;
J.Y. Yi, J.K. Kim, K.S. Soh, Phys. Rev. A 52 (1995) 3352;
J.Y. Yi, K.S. Soh, J. Korean Phys. Soc. 33 (1998) 507, quant-ph/9701002;
I.A. Pedrosa, Phys. Rev. A 55 (1997) 3219;
I.A. Pedrosa, Hadronic J. 9 (1986) 173.
[5] D.B. Monteolivia, H.J. Korsh, J.A. Nunez, J. Phys. A: Math. Gen. 27 (1994) 6897;
D.Y. Song, Phys. Rev. A 61 (2000) 102, quant-ph/9907062.
[6] J.R. Ray, J.L. Reid, J. Math. Phys. 20 (1979) 1954;
J.R. Ray, J.L. Reid, Prog. Theor. Phys. 65 (1981) 877;
J.R. Ray, J.L. Reid, J. Phys. A: Math. Gen. 13 (1980) 1969;
C. Athorne, J. Phys. A 24 (1991) 1385;
A. Karasu(Kalkanlı), H. Yıldırım, J. Nonlinear Phys. 9 (2002) 475;
P.G.L. Leach, Phys. Lett. A 158 (1991) 102;
J. Goedert, F. Haas, Phys. Lett. A 239 (1998) 348;
R.S. Kaushal, H.J. Korsh, J. Math. Phys. 22 (1981) 1904;
R.S. Kaushal, Int. J. Theor. Phys. 37 (1998) 1793;
A. Matzkin, J. Phys. A: Math. Gen. 34 (2001) 7833.
[7] F. Haas, cond-mat/0211353;
J.R. Ray, Phys. Rev. D 20 (1979) 2632.
[8] J.R. Ray, Phys. Rev. D 20 (1979) 2632.
[9] R.M. Hawkins, J.E. Lidsey, Phys. Rev. D 66 (2002) 03523, astro-ph/0112139.
[10] A.M. Gonchurenko, Y.A. Logvin, A.M. Samson, P. Shapulov, Opt. Commun. 81 (1991) 225;
A.M. Gonchurenko, Y.A. Logvin, A.M. Samson, P. Shapulov, Phys. Lett. A 160 (1991) 138.
[11] C. Athorne, J. Differential Equations 100 (1992) 82.
[12] F. Calogero, J. Math. Phys. 12 (1971) 419.
[13] F. Haas, J. Goedert, math-ph/0211032;
F. Haas, J. Goedert, Phys. Lett. A 279 (2001) 181;
J. Cervero, J. Lejarreta, Phys. Lett. A 156 (1991) 201;
A.B. Nassar, J. Math. Phys. 27 (1986) 755;
S. Moyo, P.G.L. Leach, J. Phys. A: Math. Gen. 35 (2002) 5333;
J.K. Kim, S.P. Kim, J. Korean Phys. Soc. 28 (1995) 7.
[14] W. Paul, Rev. Mod. Phys. 62 (1990) 531.
[15] F. Diedrich, J.C. Bergquist, W.M. Itano, D.J. Wineland, Phys. Rev. Lett. 62 (1989) 403;
D.J. Heinzen, D.J. Wineland, Phys. Rev. A 42 (1990) 2977.
[16] G.S. Agarwal, S.A. Kumar, Phys. Rev. Lett. 67 (1991) 3665.
[17] O.I. Drivotin, D.A. Ovsyannikov, in: Proceedings of 1999 Particle Accelerator Conference, New York, 1999, pp. 1857–1859.
[18] M. Shahinpoor, J.L. Nowinski, Int. J. Nonlin. Mech. 6 (1971) 193.
[19] Yu.B. Gaididei, K.O. Rasmussen, P.L. Christiansen, Phys. Rev. E 52 (1995) 2951;
K.O. Rasmussen, Yu.B. Gaididei, O. Bang, P.L. Christiansen, Math. Comp. Sim. 40 (1996) 339;
P.L. Hansen, D. Svensk, V.A. Pargesian, R. Podgornik, Phys. Rev. E 60 (1999) 1956.
[20] F. Finelly, G.P. Vacca, G. Venturi, Phys. Rev. D 58 (1998) 103514;
C. Bertoni, F. Finelly, G. Venturi, Phys. Lett. A 237 (1998) 331;
F. Finelly, A. Gruppuso, G. Venturi, Class. Quantum. Grav. 16 (1999) 3923.
[21] H. Rosu, P. Espinoza, M. Reyes, Nuovo Cimento B 114 (1999) 1435.
[22] V. De Alfaro, S. Fubini, G. Furlan, Nuovo Cimento A 34 (1974) 569.
[23] V. Akulov, A. Pashnev, Teor. Mat. Fiz. 56 (1983) 344 (in Russian).
[24] S. Fubini, E. Rabinovici, Nucl. Phys. B 245 (1984) 17.

[25] A. Pashnev, Sov. J. Theor. Math. Phys. 69 (1986) 1172.
[26] E. Ivanov, S. Krivonos, V. Leviant, J. Phys. A: Math. Gen. 22 (1989) 4201.
[27] J.A. de Azcarraga, J.M. Izquierdo, J.C. Perez Bueno, P.K. Townsend, Phys. Rev. D 59 (1999) 084015, hep-th/9810230.
[28] V. Akulov, M. Kudinov, Phys. Lett. B 460 (1999) 365, hep-th/9905070.
[29] E. Ivanov, S. Krivonos, J. Niederle, hep-th/0210196.
[30] E. Ivanov, S. Krivonos, O. Lechtenfeld, hep-th/0212303.
[31] V.P. Akulov, O. Cebecioğlu, A. Pashnev, hep-th/0301251.
[32] A.V. Galajinsky, hep-th/0302156.
[33] E. Ivanov, S. Krivonos, V. Leviant, J. Phys. A: Math. Gen. 22 (1989) 345.
[34] P. Claus, M. Derix, R. Kallosh, J. Kumar, P.K. Townsend, A. Van Proeyen, Phys. Rev. Lett. 81 (1998) 4553, hep-th/9804177.
[35] E. Ivanov, S. Krivonos, Lett. Math. Phys. 7 (1983) 523;
E. Ivanov, S. Krivonos, Lett. Math. Phys. 8 (1984) 39.
[36] E. Ivanov, S. Krivonos, J. Phys. A: Math. Gen. 17 (1984) L671.
[37] A. Pashnev, Nucl. Phys. B (Proc. Suppl.) 102 (2001) 240.
[38] J. Malinsky, V.P. Akulov, N. Abdellatif, A. Pashnev, Phys. Rev. D 67 (2003) 025008, hep-th/0206028.
[39] E. Kamke, Differentialgleihungen, Akademie-Verlag, Leipzig, 1959, Eq. (3.15).